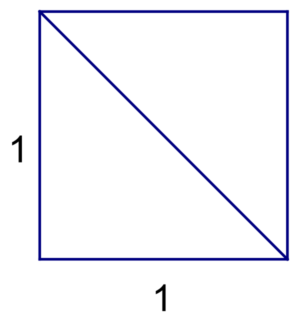
Section 9-1, Mathematics 104

Radicals and Rational Exponents

Let's start with a simple problem from geometry.

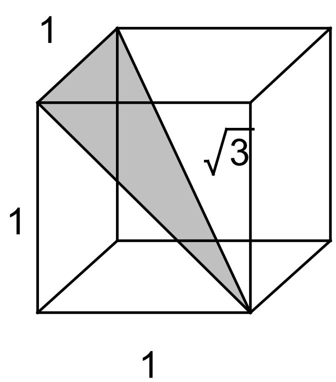
What is the length of the diagonal of a square with sides 1



We know from the Pythagorean theorem that

$$c^2 = a^2 + b^2$$
$$c^2 = 1 + 1 = 2$$
$$c = \sqrt{2}$$

For a cube we can again apply the Pythagorean theorem.



And find that the diagonal has length $\sqrt{3}$

So we find the curious fact that $\sqrt{n}$ is the length of a diagonal of an *n*-dimensional cube.

Note that the square root of any integer that is not a perfect square, such as 1, 4, 9, 16, etc. is irrational. This is also true of cubed roots and higher roots.

Definition

The square root of a number A is a solution to the equation $x^2 = A$.

Similarly the nth root of a number N is a solution to the equation $x^n = A$

Note that the square root of 4 is 2 or -2.

We call 2 the principle root of 4.

When we write $\sqrt{4}$ we mean the principle root.

If we want to indicate the negative root we can write it this way: $-\sqrt{4}$

Finding Roots

If a number is a perfect square, like 4, it is easy to find the square root.

Examples:

$\sqrt{64} =$

$\sqrt{100} =$

$\sqrt{144} =$

$\sqrt{625} =$

Cube roots and higher roots are similar

$\sqrt[3]{27} =$

$\sqrt[3]{64} =$

$\sqrt[3]{125} =$

Note that square roots of negative numbers, eg. $\sqrt{-4}$ are not real numbers, however

$\sqrt[3]{-8} = -2$

Can you see a pattern here?

Inverse Properties of roots

Note that whenever you have an nth root, you can raise it to the nth power to get the original number back.

$$\left(\sqrt[n]{a}\right)^n = a$$

If you look at a few examples, this seems obvious

$$\sqrt{5} \cdot \sqrt{5} = \left(\sqrt{5}\right)^2 = 5$$
$$\sqrt[3]{7} \cdot \sqrt[3]{7} \cdot \sqrt[3]{7} = \left(\sqrt[3]{7}\right)^3 = 7$$

Also note that

if n is odd, $\sqrt[n]{\left(a^n\right)} = a$

Examples:

$$\sqrt[5]{6^5} = 6$$
$$\sqrt[5]{(-6)^5} = 6$$

However if n is even

$$\sqrt[n]{\left(a^n\right)} = |a|$$

Example:

$$\sqrt[4]{(-2)^4} = \sqrt[4]{16} = 2$$

Rational Exponents

When dealing with exponents we have a few laws

$$a^n \cdot a^m = a^{n+m}$$
$$\left(a^n\right)^m = a^{n \cdot m}$$

This applies to integers, but shouldn't it also apply to rational numbers?

Let's explore what $5^{1/2}$ means.

$$5^{1/2} \cdot 5^{1/2} = 5^{1/2+1/2} = 5^1 = 5$$

So we have

$$5^{1/2} \cdot 5^{1/2} = 5$$

But that tell us that $5^{1/2} = \sqrt{5}$

Similarly

$$a^{1/n} = \sqrt[n]{a}$$

This suggests that in some calculations it might be easier to convert roots to exponents and use the laws of exponents.

Example:

$$\sqrt{5} \cdot \sqrt[3]{5} = 5^{1/2} \cdot 5^{1/3} = 5^{5/6} = \left(5^{1/6}\right)^5 = \left(\sqrt[6]{5}\right)^5$$

Example:

$$8^{4/3} = \left(8^{1/3}\right)^4 = 2^4 = 16$$

Example:

$$x\sqrt[4]{x^3} = x^1 \cdot x^{3/4} = x^{1+3/4} = x^{7/4}$$

Try a few of these:

$$81^{-3/4}$$

$$\left(\frac{8}{27}\right)^{3/2}$$

$$\left(\sqrt{x^3}\right)^4$$

The domain of a radical expression or a function

If you have a radical expression such as

$$\sqrt{2x-1}$$

The domain on this expression is not all real numbers.
To find the domain, find where the expression inside the radical is ≥ 0

$$2x-1 \geq 0$$
$$2x \geq 1$$
$$x \geq \frac{1}{2}$$

What is the domain of this function?

$$f\left(x\right) = \sqrt{2x+9}$$